Systematic Deceleration of Fast Modes in Multi-Scale Systems

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Multi-Scale Problems

- There exists a continuum of spatio-temporal scales without significant separation
- Dynamical Variables can be separated into Two Groups - ESSENTIAL and NON-ESSENTIAL variables
- Statistical Behavior of the ESSENTIAL variables is the main objective

In other words: We would like to integrate the equation more efficiently by sacrificing accuracy in predicting the non-essential variables

Examples:

- Molecular Dynamics
- Large-Scale structures in the Atmosphere/Ocean

Conservative Quadratically Nonlinear Systems

$$\dot{Z} = f(Z)$$

Decomposition:

$$Z = (SLOW,\ FAST) \equiv (X, Y)$$

Quadratic System:

$$\dot{X} \quad = \quad \{X, X\} + \{Y, X\} + \{Y, Y\}$$

$$\dot{Y} \quad = \quad \{X, X\} + \{Y, X\} + \{Y, Y\}$$

Conservation of Energy:

$$\frac{d}{dt}E = \frac{d}{dt}\left(X^2 + Y^2\right) = 2X\dot{X} + 2Y\dot{Y} =$$
$$\{X, X, X\} + \{X, X, Y\} + \{Y, Y, X\} + \{Y, Y, Y\} = 0$$

Previous Asymptotic Approach

(Majda, Timofeyev, Vanden-Eijnden (2006) Nonlineariy **19**)

Introduce ε:

$$\begin{aligned} \dot{X} &= \{X,X\} + \frac{1}{\varepsilon}\{Y,X\} + \frac{1}{\varepsilon}\{Y,Y\} \\ \dot{Y} &= \frac{1}{\varepsilon}\{X,X\} + \frac{1}{\varepsilon}\{Y,X\} + \frac{1}{\varepsilon^2}\{Y,Y\} \end{aligned}$$

Consider Limit

$$\varepsilon \longrightarrow 0$$

Obtain Reduced Stochastic Equation

$$dX = F(x)dt + G(x)dW$$

Computational Approach

Introduce ε:

$$\begin{aligned}\dot{X} &= \{X,X\} + \frac{1}{\varepsilon_1}\{Y,X\} + \frac{1}{\varepsilon_2}\{Y,Y\} \\ \dot{Y} &= \frac{1}{\varepsilon_1}\{X,X\} + \frac{1}{\varepsilon_2}\{Y,X\} + \frac{1}{\varepsilon_3}\{Y,Y\}\end{aligned}$$

- Consider $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 > 1$
- Values depend on the particular triad (e.g. total wavenumber)

Conservation of Energy is Preserved:

$$\dot{E} = \{X,X,X\} + \frac{1}{\varepsilon_1}\{X,X,Y\} + \frac{1}{\varepsilon_2}\{Y,Y,X\} + \frac{1}{\varepsilon_3}\{Y,Y,Y\} = 0$$

Truncated Burgers-Hopf Model (TBH)

Fourier-Galerkin Projection of

$$u_t + uu_x = 0$$

onto a finite number of Fourier modes

$$u = \sum \hat{u}_k e^{ikx}, \qquad 1 \leq |k| \leq \Lambda$$

2Λ-dimensional system of ODEs

$$\frac{d}{dt}\hat{u}_k = -\frac{ik}{2} \sum_{p+q+k=0} \hat{u}_p^* \hat{u}_q^*, \qquad \hat{u}_k^* = \hat{u}_{-k}$$

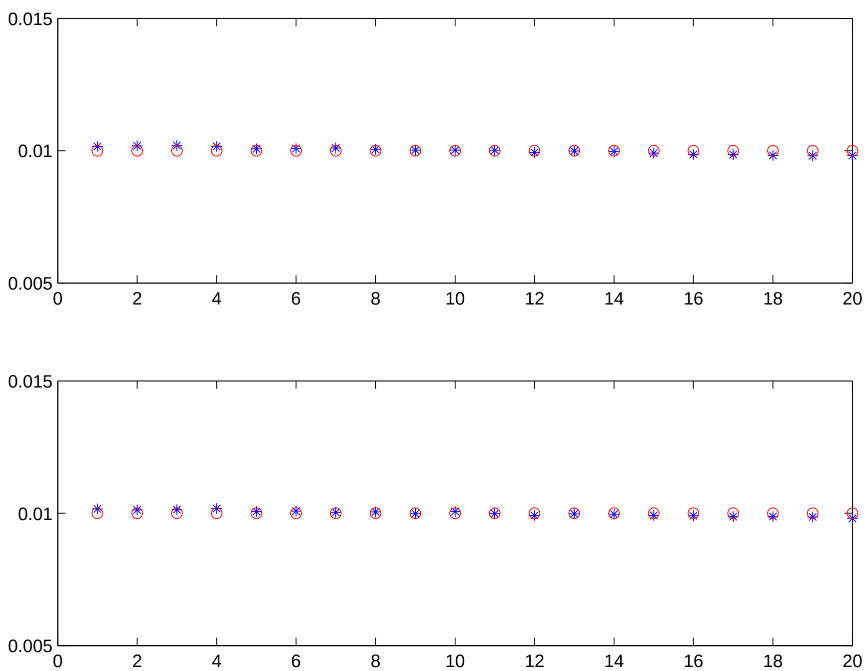
Main Features:

- Volume-Preserving Flow
- Conservation of Energy $\sum |\hat{u}_k|^2$; Equipartition
- Correlation scaling $\quad Corr.Time\{\hat{u}_k\} \sim k^{-1}$
- Gaussian distribution in the limit $\Lambda \to \infty$

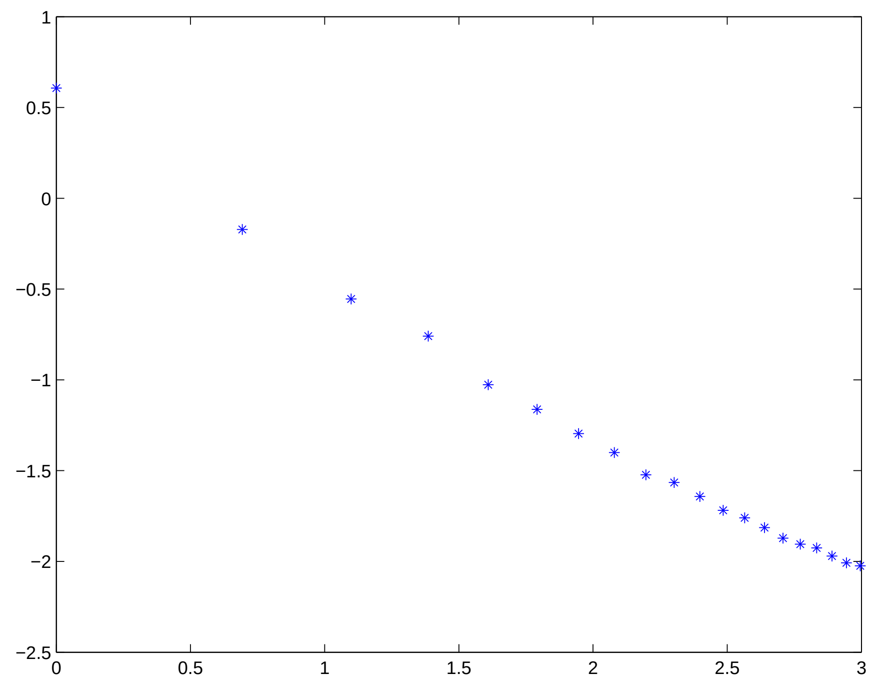
Numerical Results for TBH

$$\beta = 50, \qquad \Lambda = 20$$

Spectrum $\langle (\ \mathrm{Re}\,, \ \ \mathrm{Im}\ u_k)^2 \rangle$



Log-Log Correlation Times u_k vs k



$$\mathrm{Corr.Time} = \int \langle u_k(t) u_k(t+\tau) \rangle_t \, d\tau$$

Decelerate Triads in TBH

$$\dot{\hat{u}}_k = -\frac{ik}{2} \sum_{k+p+q=0} \hat{u}_p^* \hat{u}_q^*, \qquad 1 \le |k|, |p|, |q| \le \Lambda$$

$$\text{SLOW } = u_k, |k| \le \Lambda_1, \qquad \text{FAST } = u_k, |k| > \Lambda_1$$

Introduce $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon(n)$

$$\dot{\hat{u}}_k = -\sum_{ss} \frac{ik}{2} \hat{u}_p^* \hat{u}_q^* - \sum_{sf} \frac{ik}{2\varepsilon(n)} \hat{u}_p^* \hat{u}_q^* - \sum_{ff} \frac{ik}{2\varepsilon(n)} \hat{u}_p^* \hat{u}_q^* \,, \qquad k \le \Lambda_1$$

$$\dot{\hat{u}}_k = -\sum_{ss} \frac{ik}{2\varepsilon(n)} \hat{u}_p^* \hat{u}_q^* - \sum_{sf} \frac{ik}{2\varepsilon(n)} \hat{u}_p^* \hat{u}_q^* - \sum_{ff} \frac{ik}{2\varepsilon(n)} \hat{u}_p^* \hat{u}_q^* \,, \qquad k > \Lambda_1$$

Total Wavenumber:

$$n = |k| + |p| + |q|$$

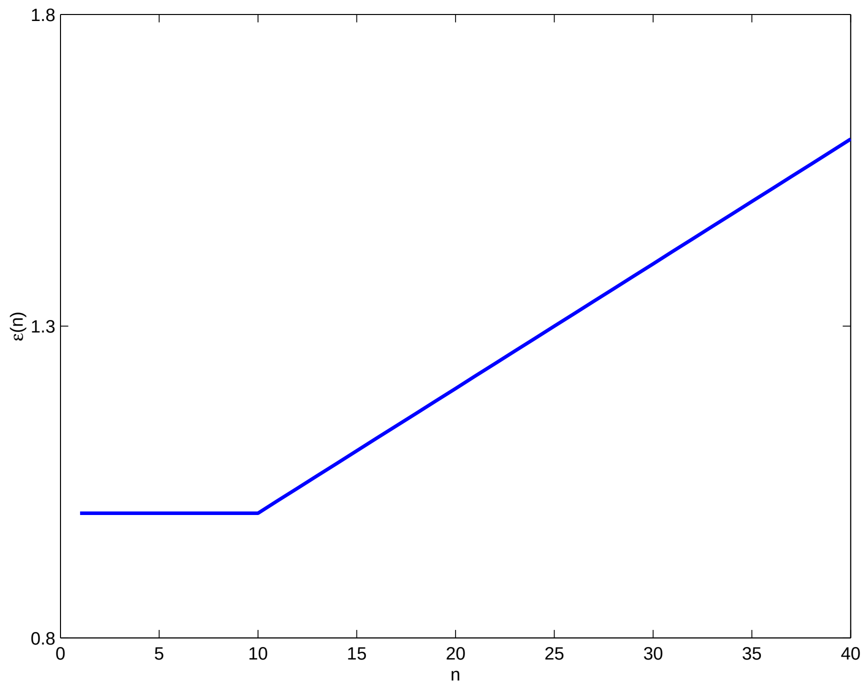
Decelerate Triads in TBH

Linear Function with slope α

$$\varepsilon(n) = \begin{cases} 1 & , \text{ if } \quad n \leq 2\Lambda_1 \\ 1 + \alpha(n - 2\Lambda_1) & , \text{ if } \quad 2\Lambda_1 < n \leq 2\Lambda \end{cases}$$

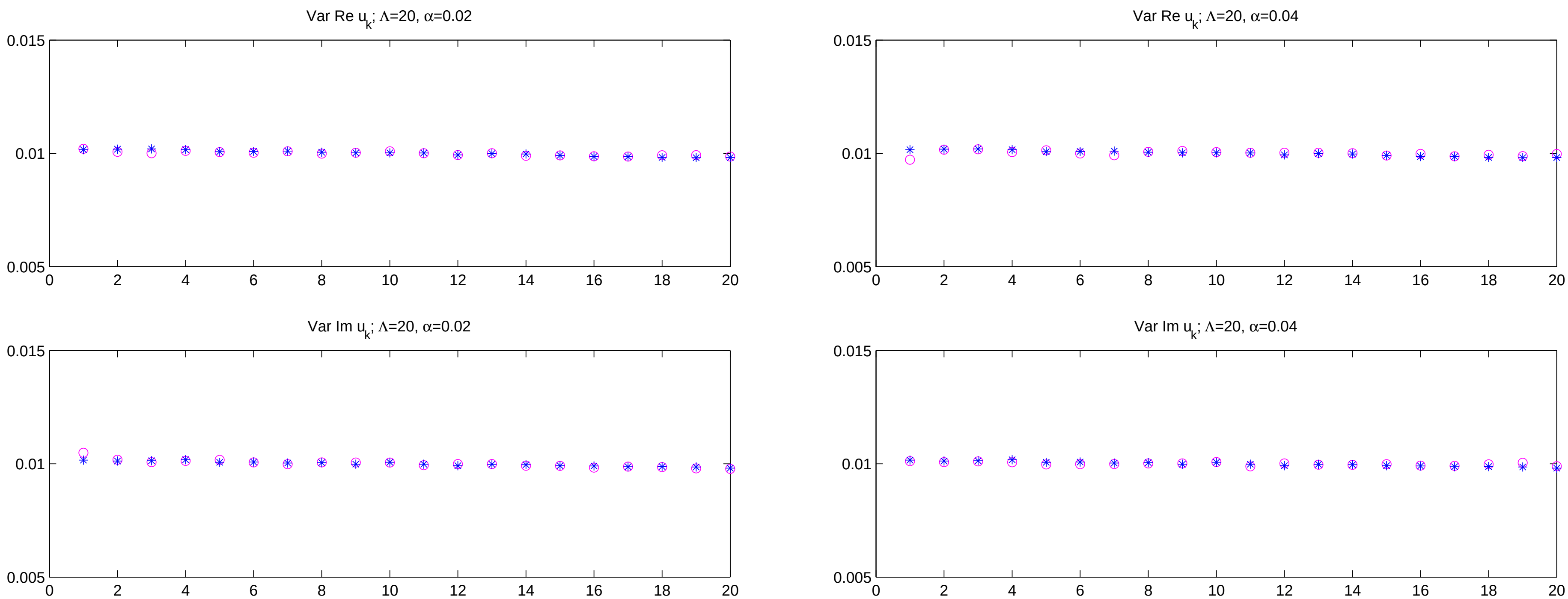
Consider: $\alpha = 0.01,\ 0.02,\ 0.04$

$\varepsilon(n)$ for $\alpha = 0.02$



Comparison of the Spectrum

$\alpha = 0.02$ $\qquad$ $\alpha = 0.04$

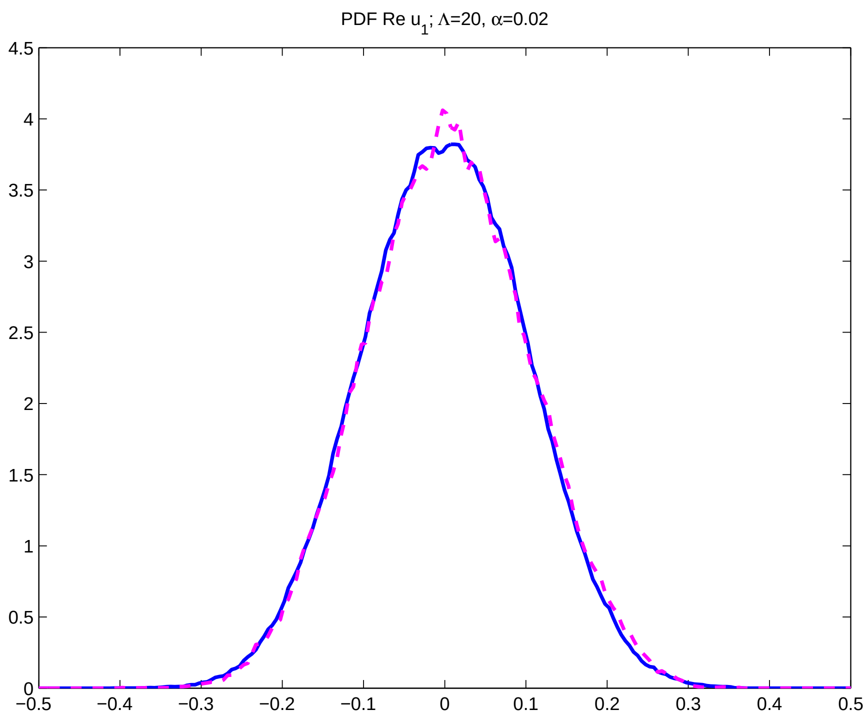


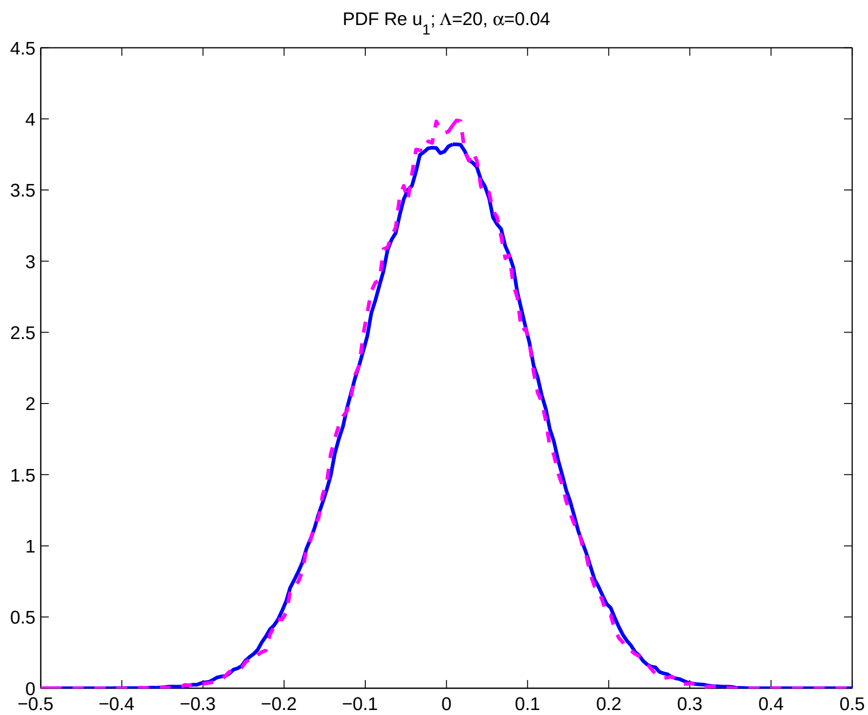
Variance of Re , Im u_k

Original Problem - Blue, Modified System - Magenta

Comparison of the PDF u_1

$\alpha = 0.02$ $\alpha = 0.04$



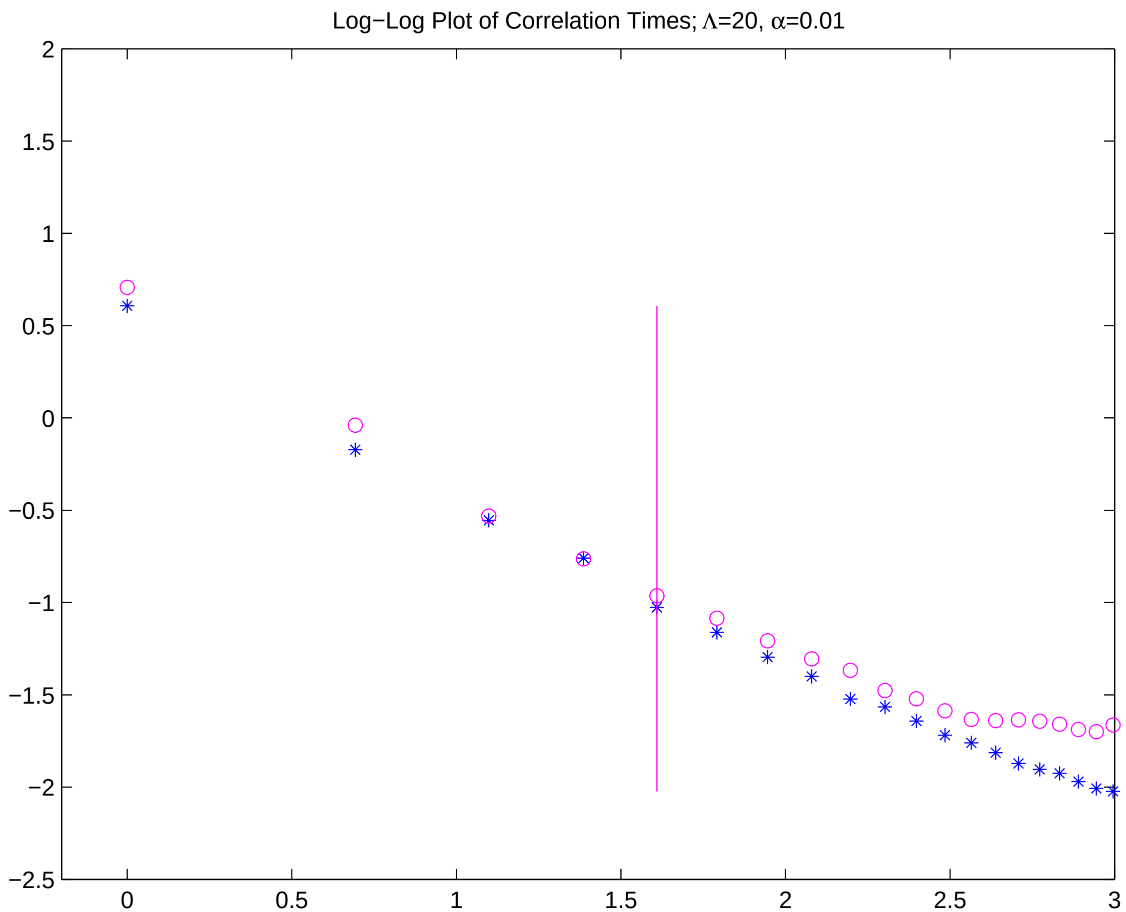


PDF of Re u_1

Original Problem - Blue, Modified System - Magenta

Correlation Times of u_k

$$\alpha = 0.01$$

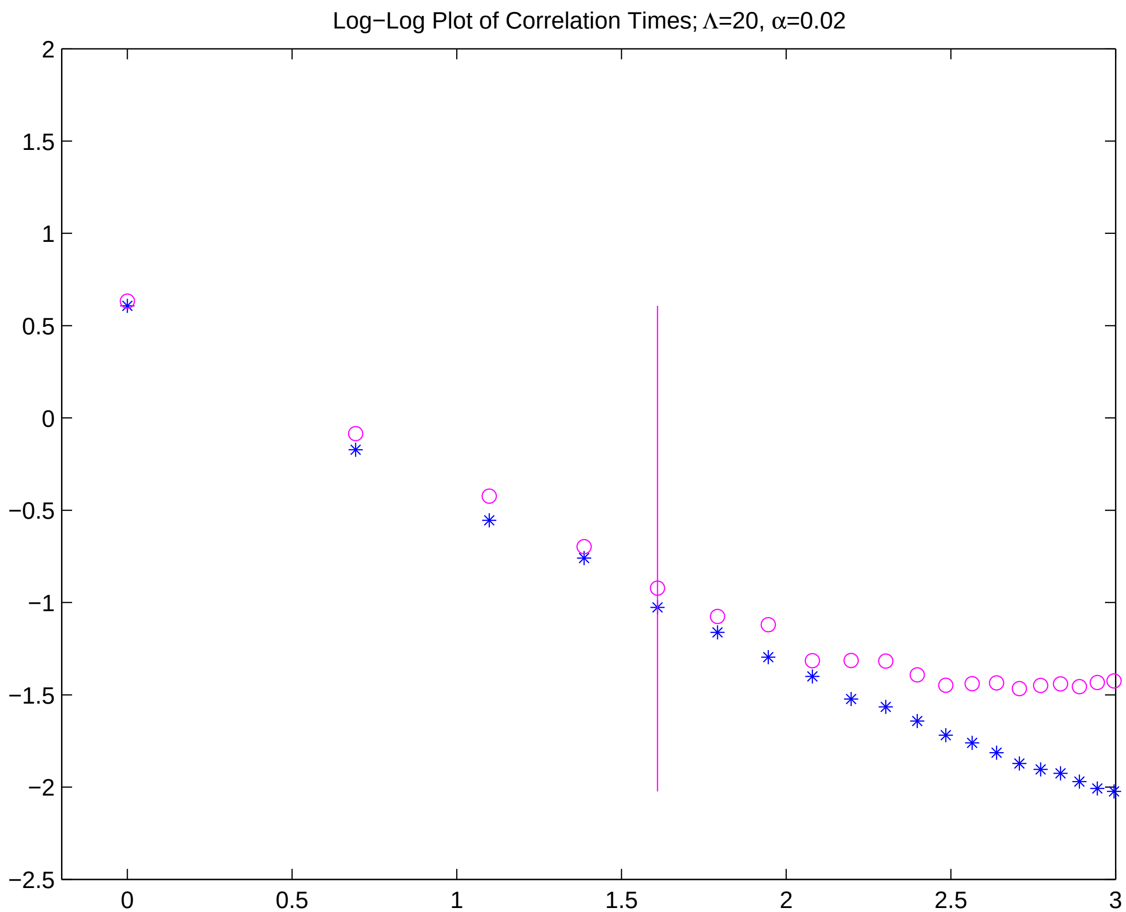


Correlation Times of Re u_k

Original Problem - Blue, Modified System - Magenta

Correlation Times of u_k

$\alpha = 0.02$

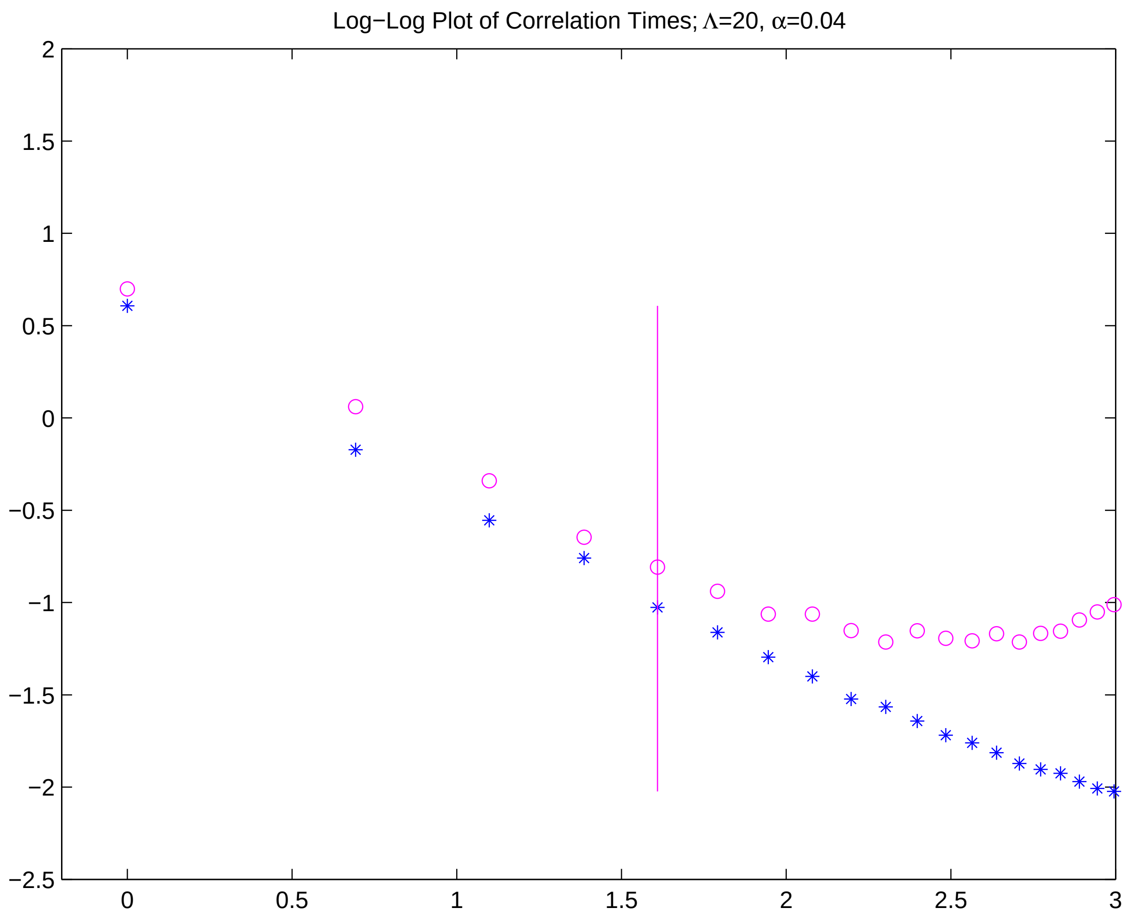


Correlation Times of Re u_k

Original Problem - Blue, Modified System - Magenta

Correlation Times of u_k

$$\alpha = 0.04$$



Correlation Times of Re u_k

Original Problem - Blue, Modified System - Magenta

Conservation of Energy in Modified Systems

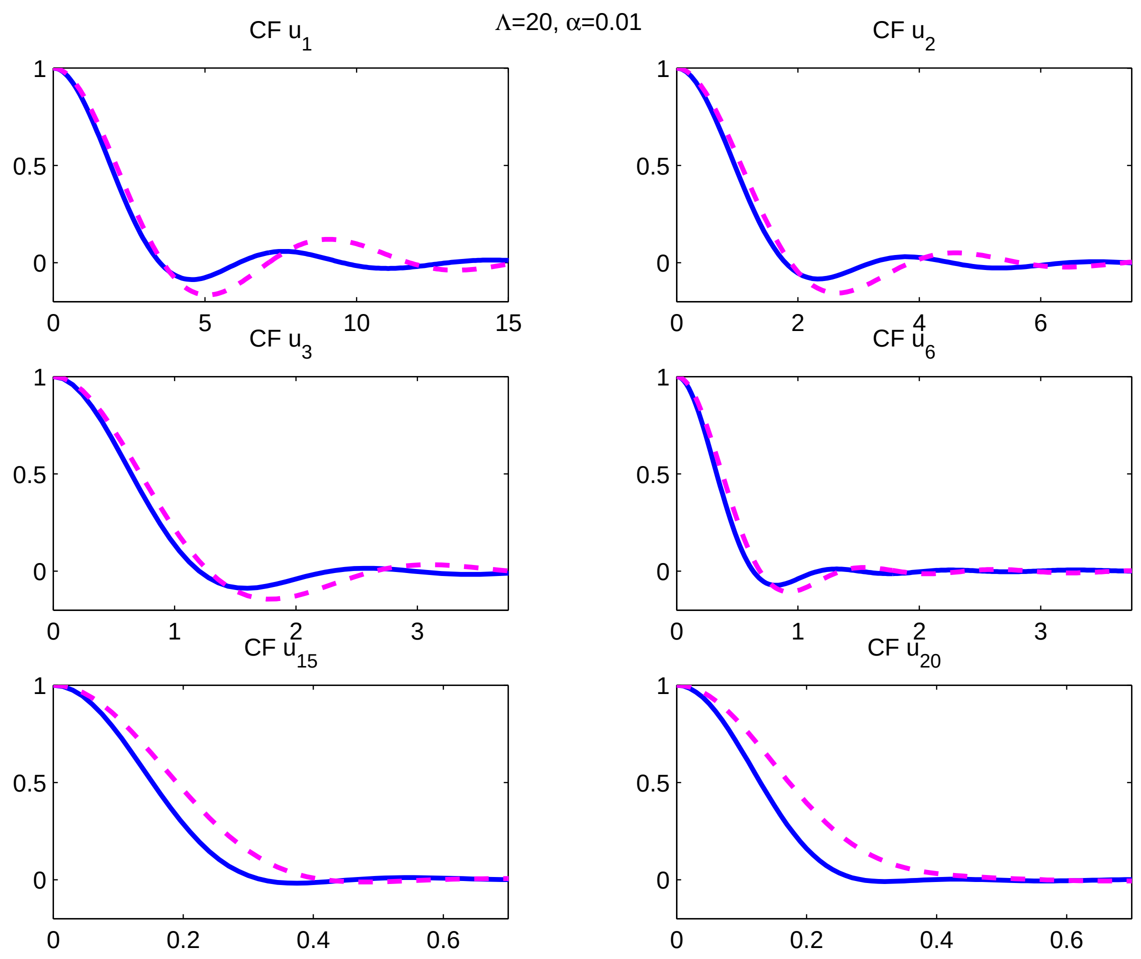
Efficiency of the Numerical Procedure:

Relative Error for the Total Energy

Δt	$\alpha = 0$ (No Modification)	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.04$
0.001	4×10^{-7}	10^{-7}	3×10^{-8}	7×10^{-9}
0.002	10^{-5}	3×10^{-6}	10^{-6}	2×10^{-7}
0.004	4×10^{-4}	10^{-4}	3×10^{-5}	7×10^{-6}

Correlation Functions of Selected Modes

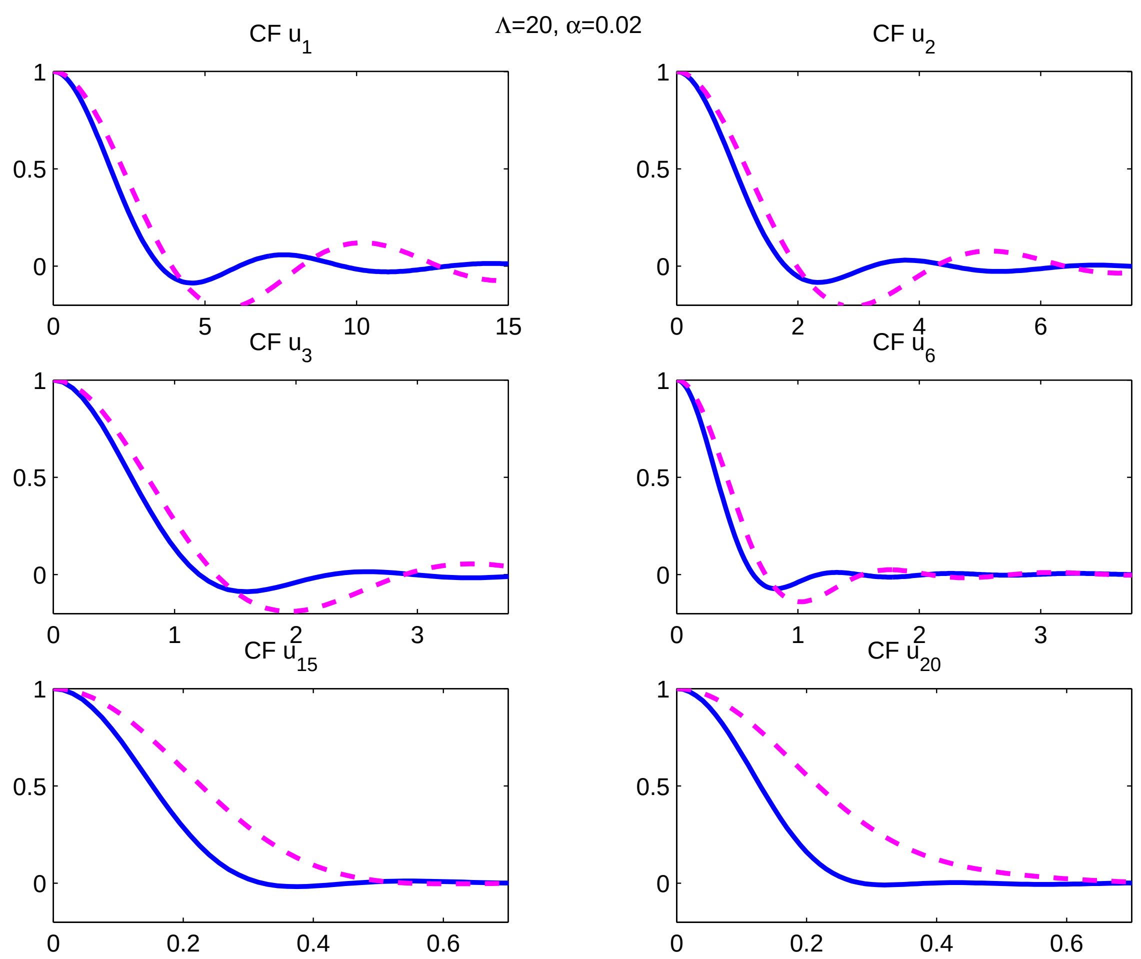
$$\alpha = 0.01$$



$$\langle u_k(t) u_k(t+\tau) \rangle_t$$

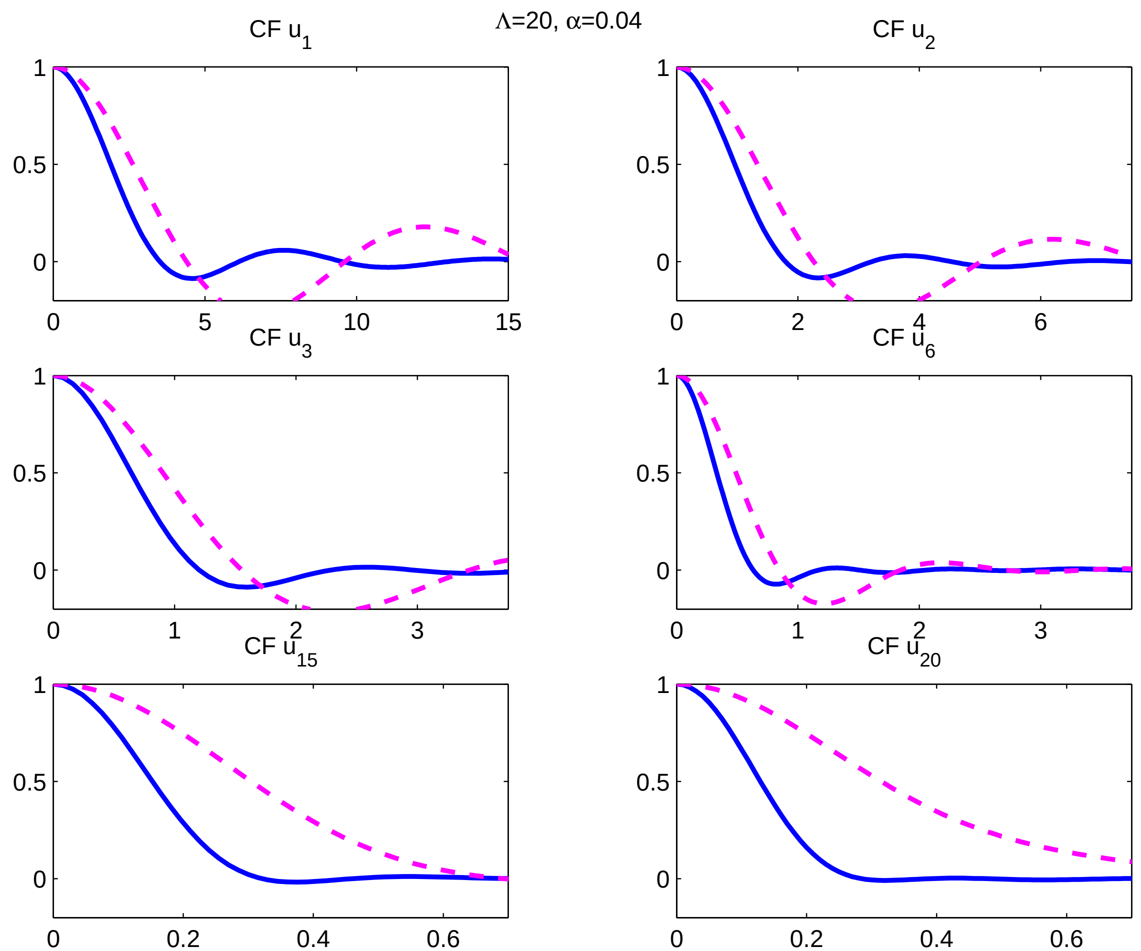
Correlation Functions of Selected Modes

$$\alpha = 0.02$$



Correlation Functions of Selected Modes

$$\alpha = 0.04$$



Conclusions

- Modify Equations to allow for bigger time-step in Direct Numerical Simulations
- Modification is Consistent with the Conservation of Energy and Volume-Preserving Flow
- Equations become "less stiff" and Energy Conservation is Improved in Simulations with bigger time-step
- Averaged Quantities are the Main Criteria for Consistency
- Energy Spectrum and Correlation Times are well-reproduced by the Modified Equations
- Slight Discrepancies in the oscillatory structure of Correlation Functions