Multi-Scale Coupling in Rotating and Stratified Flows

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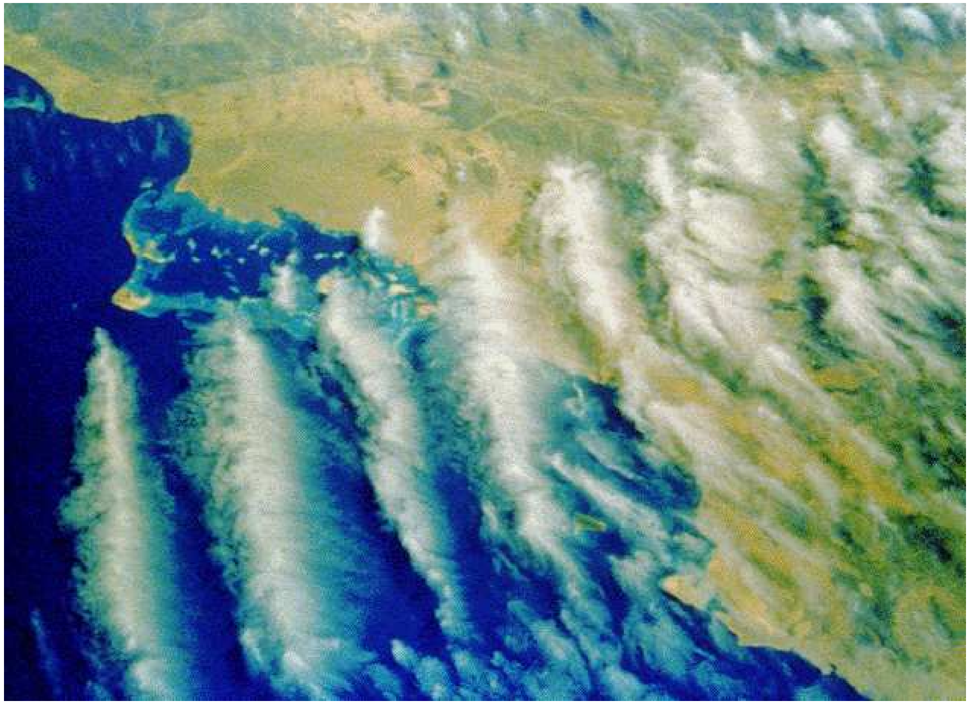
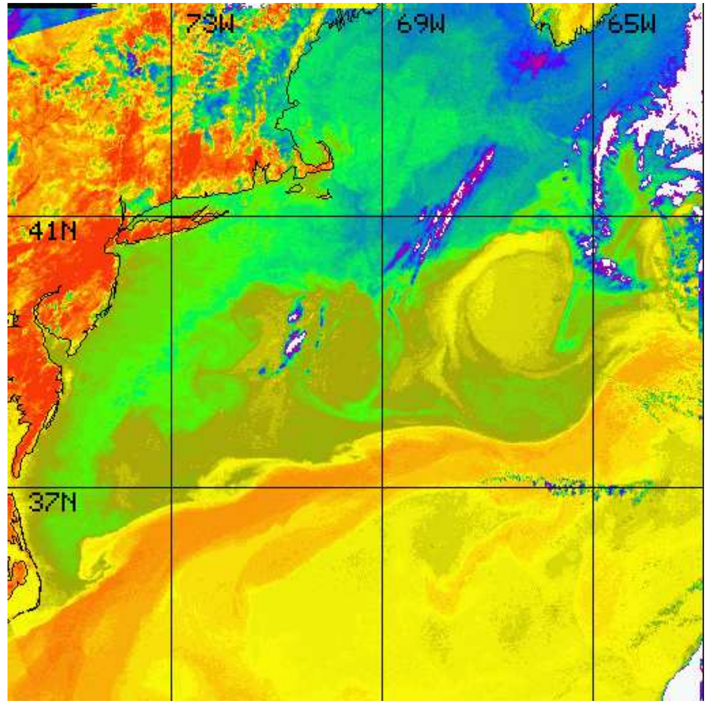




Roll clouds in the jet stream over Saudi Arabia
and the Red Sea (L) ; Gulf Stream (R)

A starting point to understand the oceans/atmosphere.

1. Do small-scale fluctuations contribute to generate large-scale, coherent structures?

JETS, LAYERS AND VORTICES

2. What is the role of wave interactions for coupling?

3. How can we construct a complete understanding of wave and vortical interactions?

Vertically stratified flow, rotating about the vertical $\hat{z}$-axis:

A common framework:

$$\frac{\partial u}{\partial t} + \frac{1}{R_*}\mathcal{L}(u) + \mathcal{N}(u,u) = \frac{1}{Re}\mathcal{D}(u) + \mathcal{F}$$

The state variable is u; $\mathcal{L}$ is skew-symmetric; $\mathcal{N}$ is quadratic

R_* is the Rossby Ro, Rhines Rh or Froude Fr number

$$Ro = \frac{U}{2\Omega L}, \quad Rh = \frac{U}{L^2\beta}, \quad Fr = \frac{U}{NL}, \quad Re = \frac{UL}{\nu}$$

$$f = 2\Omega + \beta y, \qquad \rho_{\text{total}} = \rho_o - bz + \rho, \qquad N = \left(\frac{gb}{\rho_o}\right)^{1/2}$$

Rossby, Rhines and Froude numbers in geophysical flows

Pedlosky (1986) estimates:

- $Ro \approx 0.14$ and $Rh \approx 1$ for typical synoptic-scale winds at mid-latitudes

  $U \approx 10\ \mathrm{m\ s^{-1}}$, $L \approx 1000\ \mathrm{km}$

- $Ro \approx 0.07$ and $Rh \approx 0.5$ in the western Atlantic

  $U \approx 5\ \mathrm{cm\ s^{-1}}$, $L \approx 100\ \mathrm{km}$

Typical values of N/f are $N/f \approx 100$ in the stratosphere and $N/f \approx 10$ in the oceans.

Solutions in the unforced, linear, inviscid limit

The solution form

$$u(\mathbf{x},t;\mathbf{k}) = \phi(\mathbf{k}) \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \sigma(\mathbf{k})\frac{t}{R_*}\right)\right] + \text{ c.c.}$$

with eigenmodes $\phi(\mathbf{k})$ and eigenvalues $\sigma(\mathbf{k})$.

- Wave modes $\phi_+(\mathbf{k})$ and $\phi_-(\mathbf{k})$ with $\sigma_\pm(\mathbf{k}) \neq 0$
- A non-wave (vortical) mode $\phi_0(\mathbf{k})$ with $\sigma_0(\mathbf{k}) = 0$.

Slow wave modes (as important as slow vortical modes!)

- 3D rotation-dominated flows

$$\sigma_{\pm}(\mathbf{k}) \approx \pm \frac{k_z}{k}; \quad \text{slow when} \;\; k_z = 0$$

(vertical shear layers/vortical columns)

- 2D β-plane flows

$$\sigma_{-}(\mathbf{k}) = -\frac{k_x}{k^2}; \quad \text{slow when} \;\; k_x = 0 \;\; \text{(zonal flows)}$$

- Stratification-dominated flows

$$\sigma_{\pm}(\mathbf{k}) \approx \pm \frac{k_h}{k}; \quad \text{slow when} \;\; k_h = 0 \;\; \text{(horizontal layers = VSHF)}$$

Linear eigenmode representation for nonlinear flows

Since $\boldsymbol{\phi}_s(\mathbf{k})$, $s = \pm, 0$ form an orthogonal basis

$$u(\mathbf{x},t) = \sum_{\mathbf{k}} \sum_{s} b_s(t;\mathbf{k}) \boldsymbol{\phi}_s(\mathbf{k}) \exp\left[i \left(\mathbf{k}\cdot\mathbf{x} - \sigma_s(\mathbf{k}) \frac{t}{R_*} \right) \right]$$

and the equations become

$$\left(\frac{\partial}{\partial t} + \frac{1}{Re} k^2 \right) b_{s_{\mathbf{k}}}(t;\mathbf{k})$$

$$= \sum_{\triangle_{\mathbf{k},\mathbf{p},\mathbf{q}}} \sum_{s_{\mathbf{p}},s_{\mathbf{q}}} C^{s_{\mathbf{k}},s_{\mathbf{p}},s_{\mathbf{q}}}_{\mathbf{k},\mathbf{p},\mathbf{q}} b^*_{s_{\mathbf{p}}}(t;\mathbf{p}) b^*_{s_{\mathbf{q}}}(t;\mathbf{q}) \exp\left[i \left(\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}} \right) \frac{t}{R_*} \right]$$

up to 27 interaction types, including 3-wave interactions

PDE Reduced Models

PDE Reduced Models result from restriction of the sum to any subset of interactions

Energy is quadratic and conserved by the truncation

Example

Boussinesq Slow Vortical (SV) interactions = 3D QG

Embid & Majda ’96, ’98, Babin et al. ’02, SW ’02

Exact resonances dominate for $R_* \to 0$:

$$\left(\frac{\partial}{\partial t} + \frac{1}{Re}k^2\right) b_{s_{\mathbf{k}}}(t;\mathbf{k})$$

$$= \sum_{\triangle_{\mathbf{k},\mathbf{p},\mathbf{q}}} \sum_{s_{\mathbf{p}},s_{\mathbf{q}}} C_{\mathbf{k},\mathbf{p},\mathbf{q}}^{s_{\mathbf{k}},s_{\mathbf{p}},s_{\mathbf{q}}} b^*_{s_{\mathbf{p}}}(t;\mathbf{p}) b^*_{s_{\mathbf{q}}}(t;\mathbf{q}) \exp\left[i\left(\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}}\right)\frac{t}{R_*}\right]$$

Exact resonances dominate for $R_* \to 0$

$$\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}} = 0$$

Different types of exact resonances

- 3D Boussinesq

(SV, SV, SV) is 3D QG: dominate for $Ro \sim Fr \to 0$

Others: (SV, F, F), (F, F, F)?

- Stratification dominated flow with $Fr \to 0$

(SW, SW, SW) VSHF dominate Embid & Majda (1998)

- Rotation dominated flow with $Ro \to 0$

(SW, SW, SW) are purely 2D interactions

Others: (SW, F, F), (F, F, F)?

No coupling of fast & slow modes by exact resonances!

- (SV, F, F) and (SW, F, F)

exact resonances cannot transfer energy from fast waves into slow modes

Longuet-Higgins & Gill (1967), Greenspan (1969), Phillips (1968), Warn (1986), Lelong & Riley (1991), Bartello (1995), Embid & Majda (1996, 1998)

Physical and Numerical Experiments at finite, small R_*

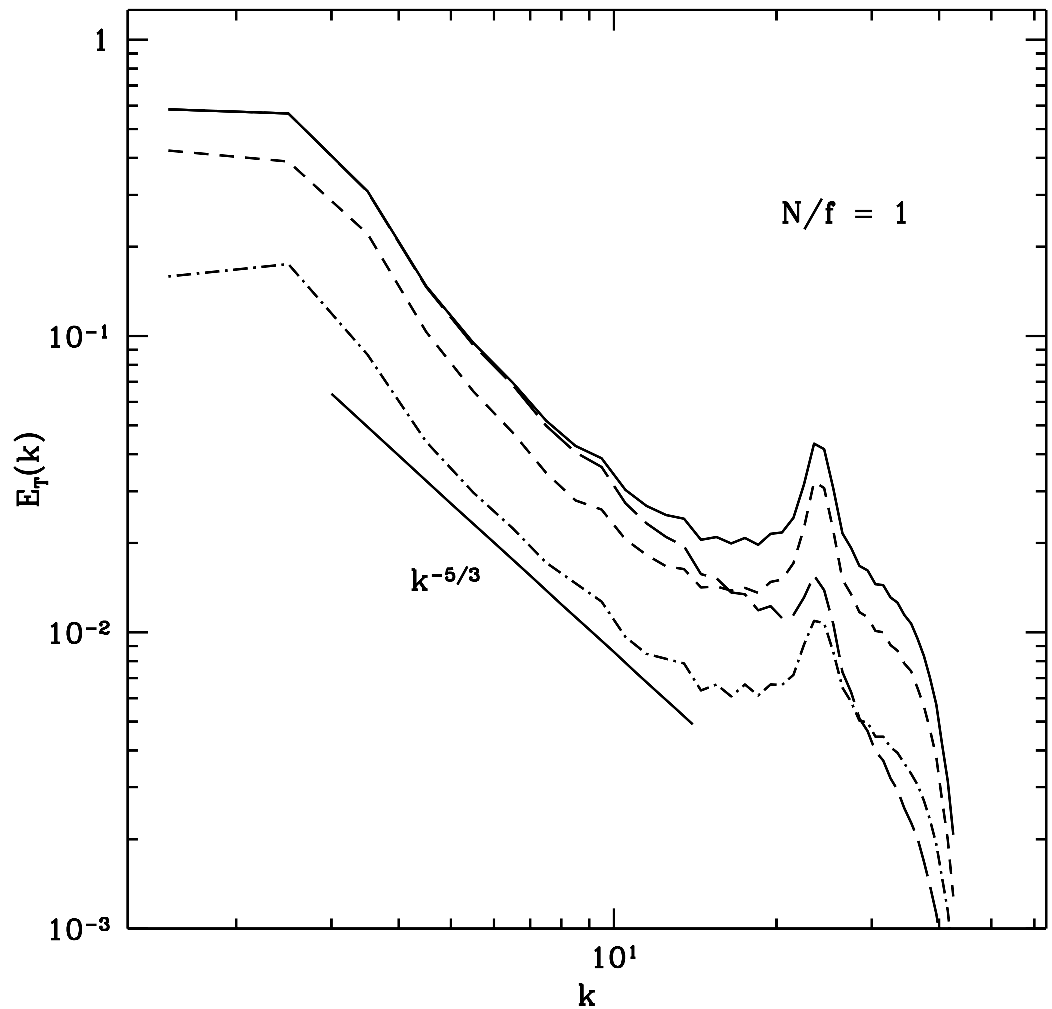
- For $Ro = Fr$ finite small, the flow is near 3D-QG

- Otherwise, small-scale fluctuations self-organize into large-scale, Slow Wave modes

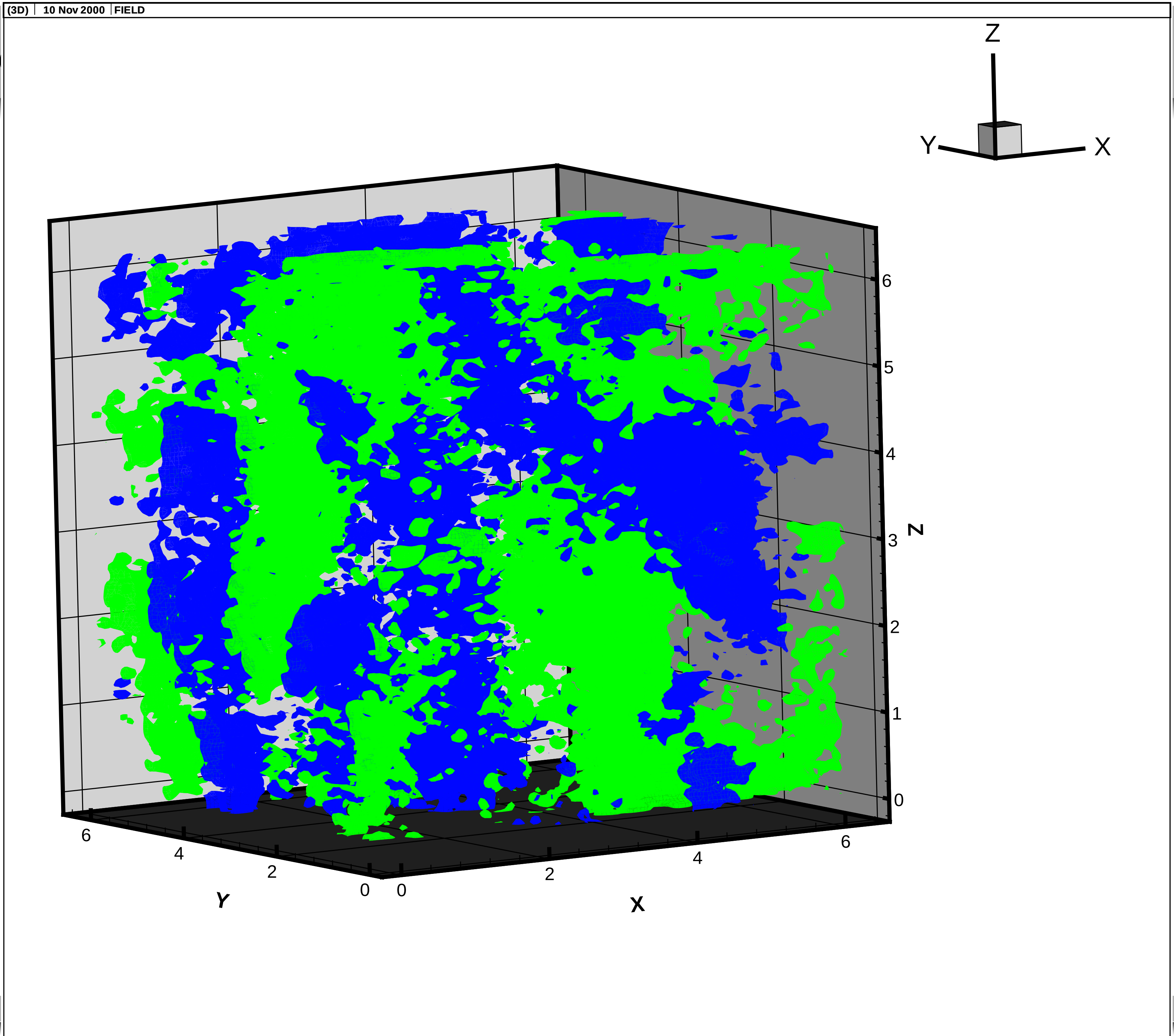
  - VSHF in stratification-dominated flows
  - 2D vortical columns for 3D rotation-dominated flows
  - zonal flows on the β-plane

How does energy get into the Slow Waves? (not exact resonances)

Boussinesq: Fr = Ro = 0.2; Full flow is close to 3D QG



Solid: $E_T(k)$; Long dash: $E_{PV}(k)$; Dash: $K(k)$; Dot-dash: $P(k)$


(3D) 10 Nov 2000 FIELD
Z
Y
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Z
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Y
0
2
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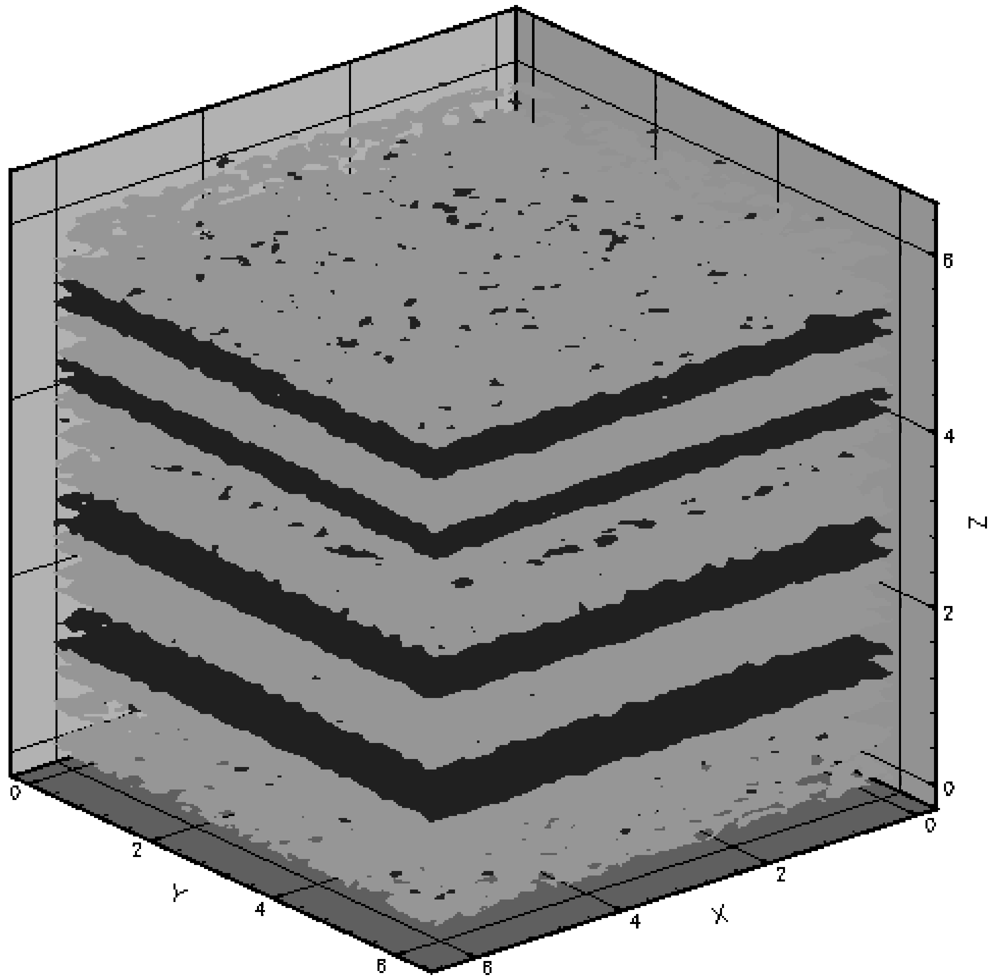
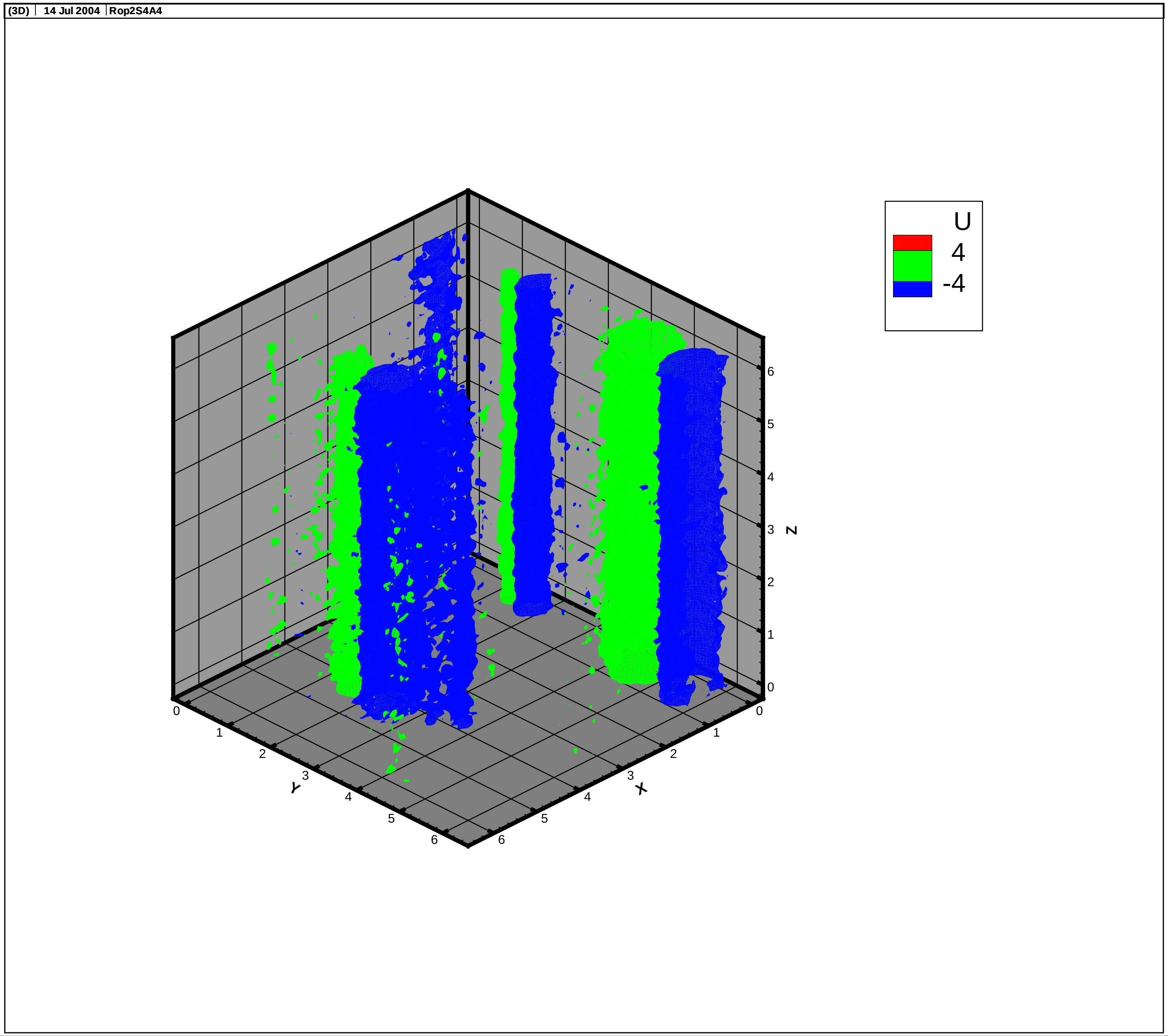
6
X

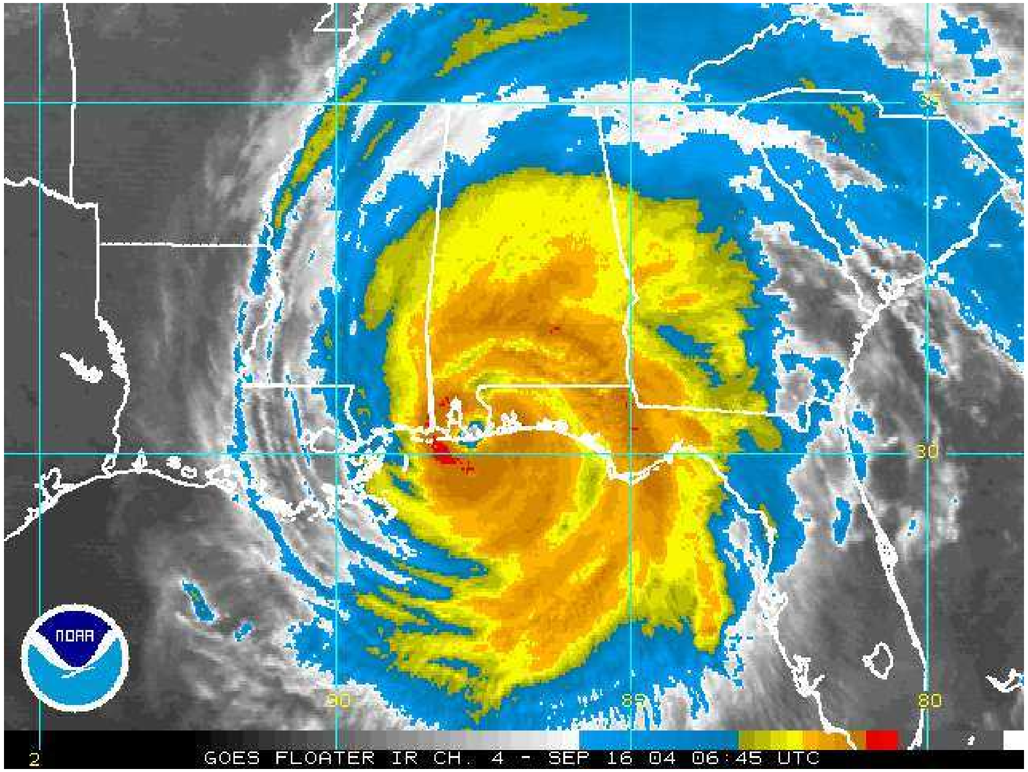
Strongly stratified flow with $Fr = 0.2$, $N/f = 100$

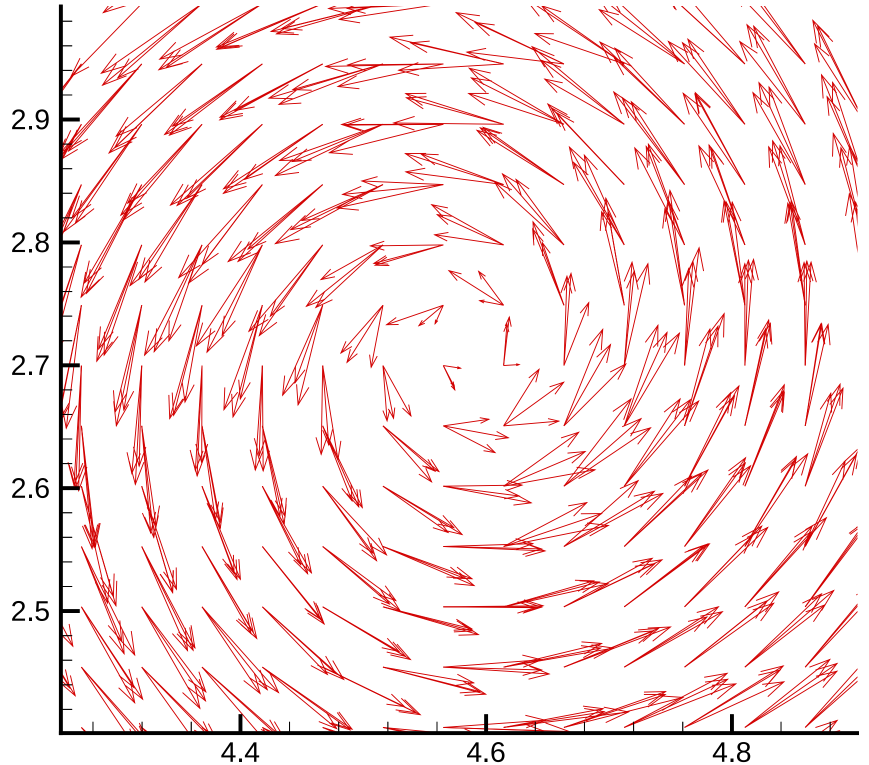


Cyclonic vortices in pure rotation with $Ro = 0.085$



Cyclonic vortices in nature





Left: Hurrican Ivan hits Florida. Right: An overlay of velocity vectors in three planes from a 128^3 simulation with random forcing at small scales. In both cases, large-scale, cyclonic vortices are fueled by smaller-scale fluctuations.

Near Resonances for finite small R_*

Are near resonances responsible for the formation of jets, layers and vortices?

Near-resonant interactions are defined by

$$|\sigma(\mathbf{k}) + \sigma(\mathbf{p}) + \sigma(\mathbf{q})| = O(R_*)$$

and become important on time scales $O(1/R_*)$ (Newell, 1969)

Reduced models of near/non-resonances

- Interactions among near-resonances only, with

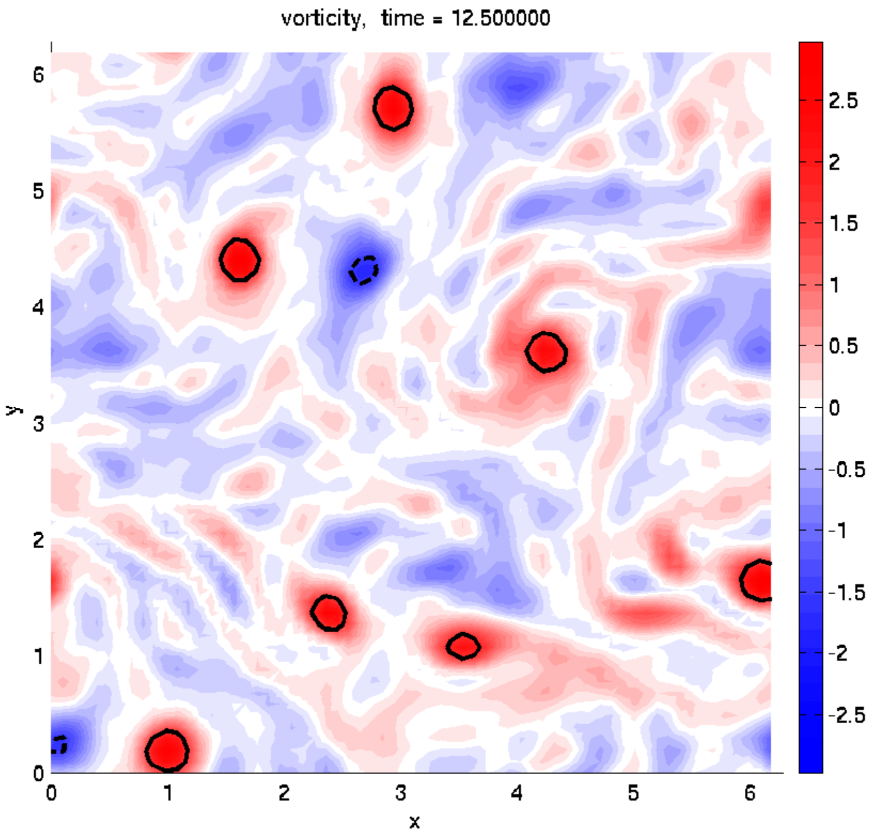
$$|\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}}| \leq R_*$$

- Interactions among non-resonances only, with

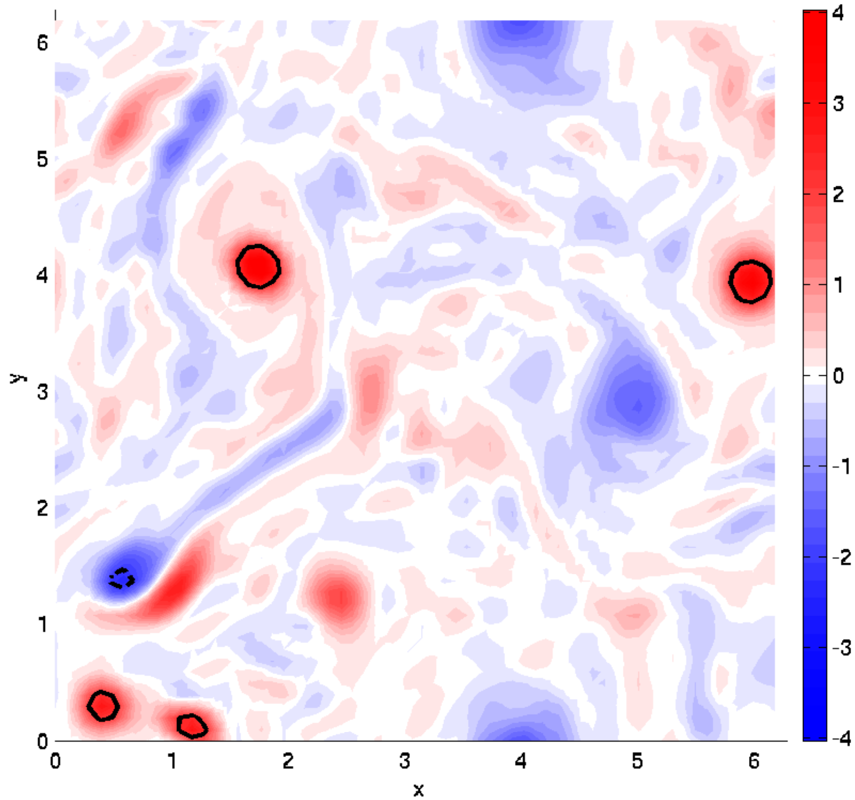
$$|\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}}| > R_*$$

- Do not correspond to PDEs
- FFTs can no longer be used $==>$ low resolution!

$\hat{z}$-averaged vertical vorticity for 3D rotation with $Ro = 0.085$

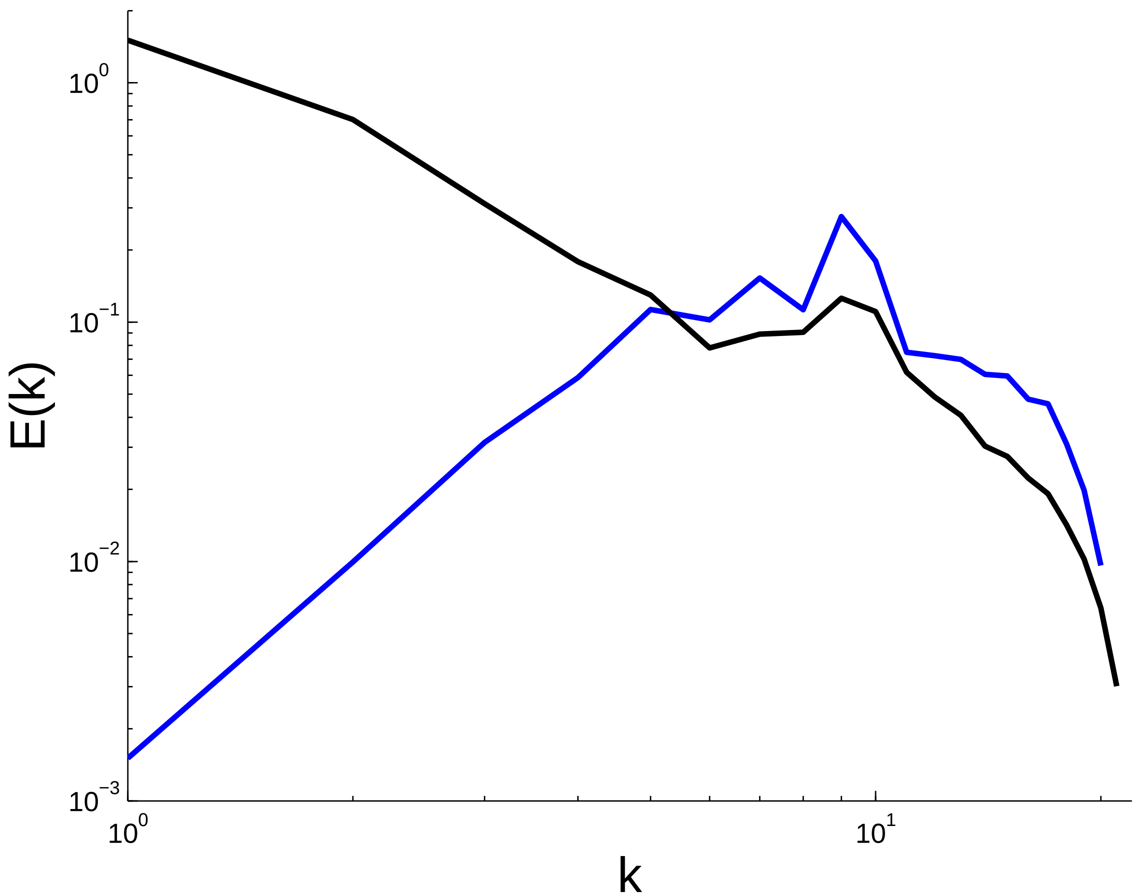


Full



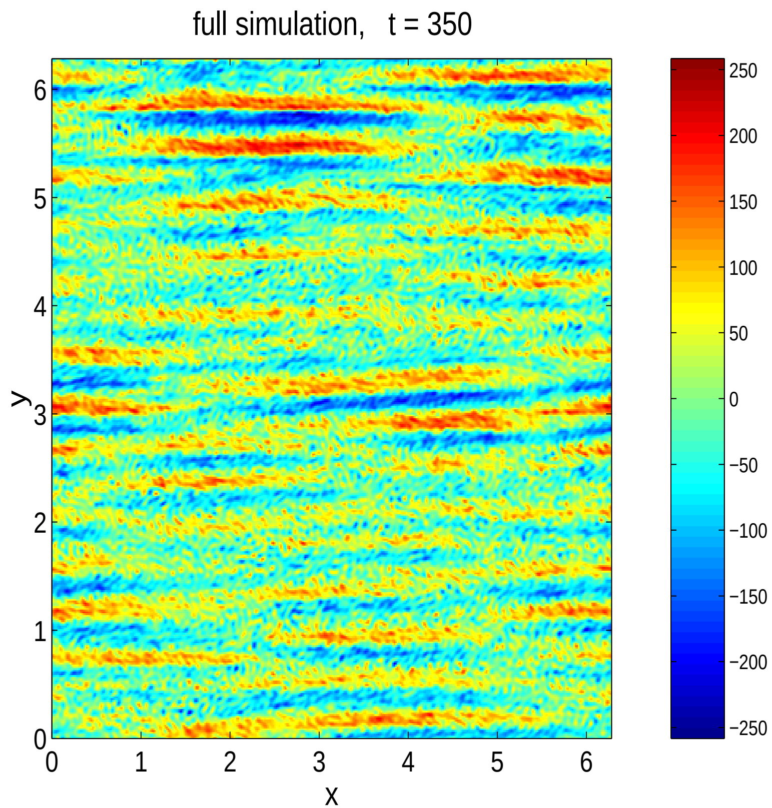
Near resonances (12%)

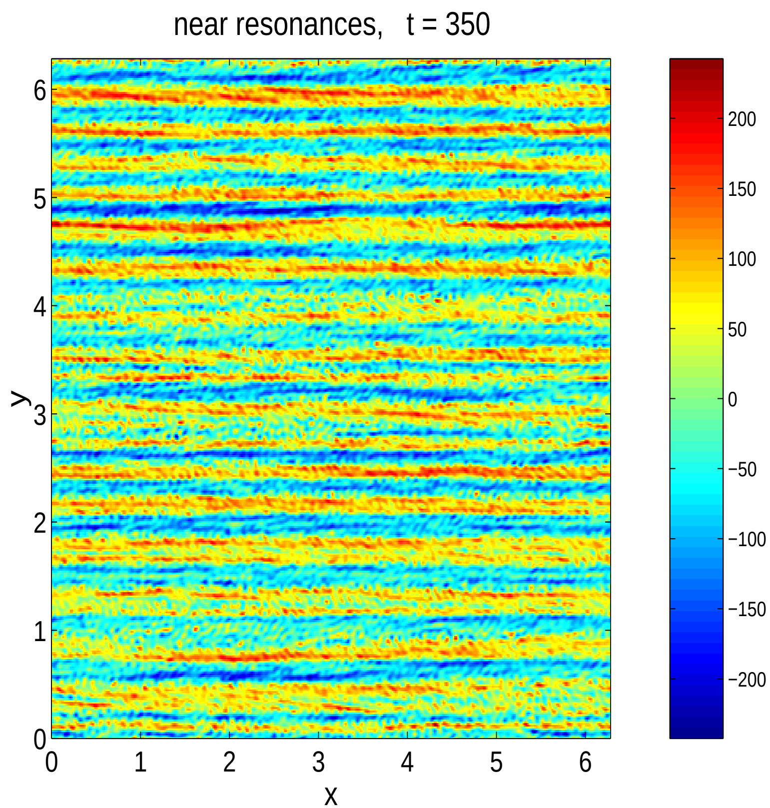
Non-resonances for $Ro = 0.085$



Black: full; Blue: Non-resonances ($\approx 90\%$)

Vorticity on the β-plane with $Rh = 0.5$





Full

Near resonances (33%)

Which near resonances matter most?

e.g. in pure rotation: (SW,F,F) or (F,F,F) ?

Pure 2D flow is the interaction of (SW,SW,SW)
and has no cyclone/anti-cyclone asymmetry

PDE models keeping (SW,SW,SW) and (SW,F,F) or (F,F,F) :

- keep some near-resonances
- can exhibit asymmetry
- may be computationally efficient

Stay tuned for results by Li Wang!

New PDE Reduced Models for a complete understanding of wave and vortical interactions

Rotating Shallow Water Flow: $\sigma(\mathbf{k}) = \pm(f^2 + gh_o k^2)^{1/2}$

Three linear eigenmodes: SV, $\pm$ F (no SW)

(SV, SV, SV) is 2D QG flow (Salmon)

Proof of Concept in RSW flow

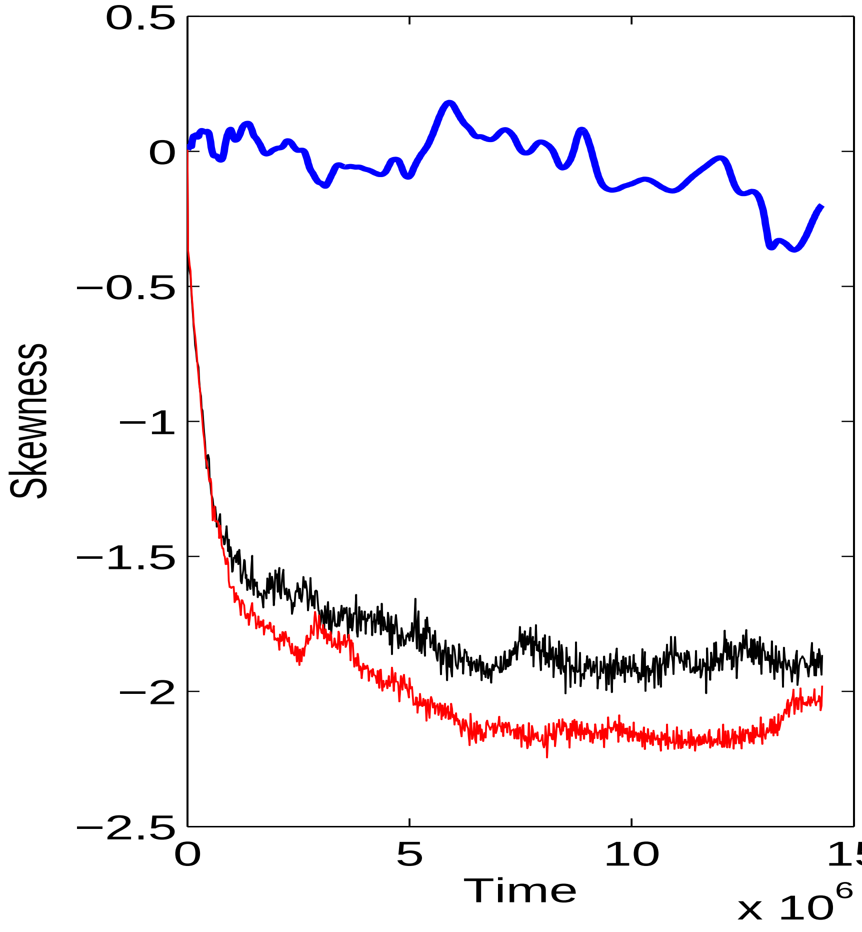
(SV,SV,SV) is 2D QG with no asymmetry between cyclones and anticyclones

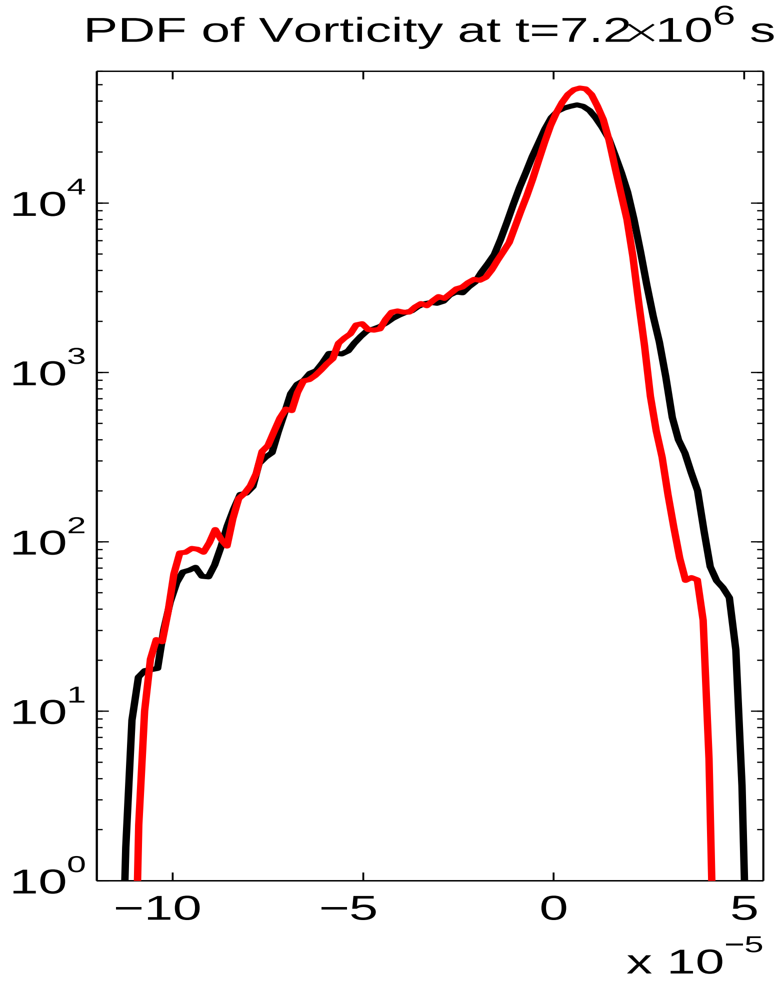
PPG: 2D QG + (SV, SV, F) + (F, SV, SV)

P2G: PPG + catalytic (F, SV, F)

Is there asymmetry in PPG and/or P2G?

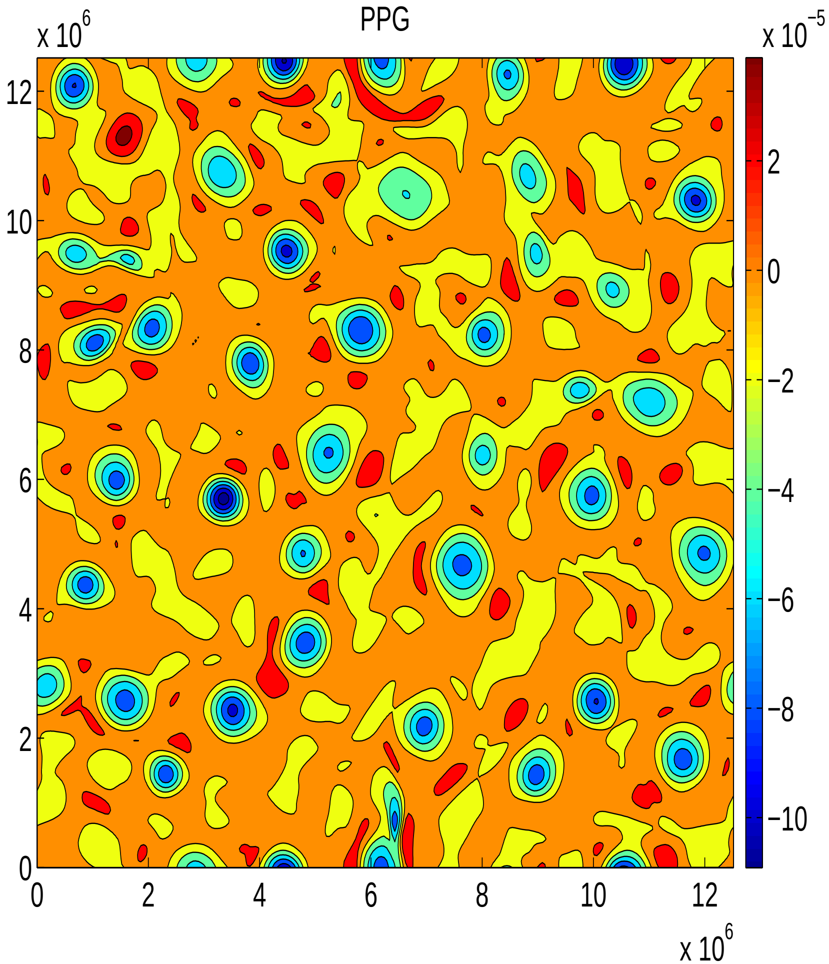
Decay from unbalanced I.C.s: Fr = 0.25, Ro = 0.4 (Polvani et al. 1994)



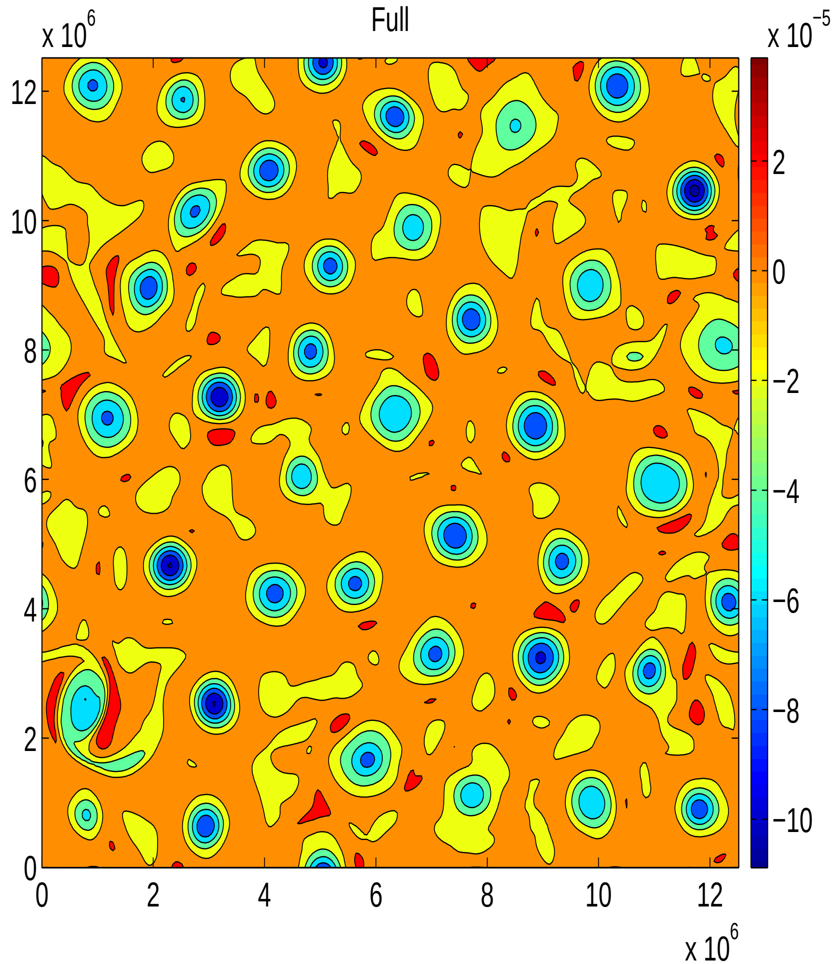


Black: PPG, RED: Full RSW, Blue: QG

RSW flow with $Fr = 0.25$, $Ro = 0.4$



PPG



Full RSW

Conclusions

A path to understand all wave-vortical interactions in dispersive systems

- Restrict the wave-space sum to include any subset of different interactions
- Inverse transform to derive a PDE in physical space
- Use numerical simulations to compare the reduced PDE to the full equations
- Some reduced PDEs may be more anemable to rigorous analysis than the full PDEs